

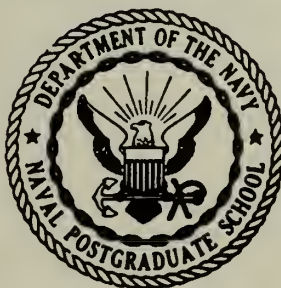
Harold A. Titus

OPTIMAL CONTROL SYSTEMS, PART II  
(SYNTHESIS).

TA7  
.U6  
no.38

Library  
U. S. Naval Postgraduate School  
Monterey, California

# UNITED STATES NAVAL POSTGRADUATE SCHOOL



## OPTIMAL CONTROL SYSTEMS Part II (Synthesis)

by

Harold Titus  
James Demetry

~~Technical Report~~

*Research Paper*

No. 38  
September 1963

Supported in Part by the Office of Naval Research



## ABSTRACT

This paper discusses briefly the digital and analog synthesis of several optimum and quasi-optimum minimum-time controls for systems with real, null, and complex eigenvalues. Controls are designed using nonlinear feedback. Examples demonstrate the simplicity of the design.

An example of the control of a nonlinear system is given using a technique which is effectively a linearization of the system about each state point on a trajectory. A quasi-optimum minimum-time control is generated by substituting the nonlinear functions of the states for their respective linearized characteristics.



Optimal control systems, Part II (Synthesis

THE SYNTHESIS OF OPTIMUM AND QUASI-OPTIMUM MINIMUM  
TIME CONTROLS FOR SECOND ORDER SYSTEMS

by

A.  
Harold Titus  
James Demetry

This Research was conducted  
at the (United) (States) Naval Postgraduate School  
with the support of the Office of Naval Research.

TAT.

U6

no. 38



## I INTRODUCTION

The purpose of this paper is to demonstrate to the practicing engineer the simplicity of synthesis of several minimum-time controls. The emphasis herein is to move quickly from the theory in bang-bang controls to the task of generating the control functions in terms of the state variables or feedback variables.

The task is to design a physically constrained control which will drive the state vector from some initial condition to the origin of the state space in minimum time.

Simple controls using nonlinear feedback are demonstrated. In all of the examples shown, the problems were simulated on both digital and analog computers. One of the examples represented the control of a motor with a digital computer.



## 1.1 Mathematical Development

Pontryagin's Maximum Principle [1] enables one to show that a minimum time control acts at its maximum effort whenever the measurement of the error is above the noise level.

The system to be considered may be described in general form as

$$\dot{\underline{x}} = F\underline{x} + D\underline{u} \quad (1)$$

or

$$\dot{\underline{x}} = f(\underline{x}, u) \quad (1')$$

We will restrict ourselves to linear systems with nonlinear saturation-type control. That is:  $F$  is a constant ( $n \times n$ ) matrix,  $\underline{x}$  is an  $n$ -vector representing the error states of the system,  $D$  is a constant vector, and the control,  $u$ , is a scalar and is constrained ( $|u| \leq N$ ).

From the Maximum Principle we see that  $u$  will operate at its maximum values. This is easily seen by noting that  $u$  appears only linearly in the Hamiltonian:

$$H = \sum_i p_i f_i \quad (\text{maximum}) \quad (2)$$

where the  $p_i$  are the adjoint variables given by



$$\dot{\underline{p}} = -F^t \underline{p} \quad (3)$$

and the  $f_i$  are the rows of (1). To obtain the maximum of  $H$  with respect to  $u$  requires that

$$u = N \operatorname{sgn} (\sum d_i p_i) \quad (4)$$

One may avoid the task of determining the adjoint variables by generating the control as a function of the states. This then gives simply a feedback control.

A heuristic argument for generating these control functions follows. We have an initial value-final value extremal problem with a cost function and a set of differential equations of constraint. Starting at some initial state  $\underline{x}(0) = \underline{C}$ , we desire to reach  $\underline{x}(T) = \underline{0}$  in minimum time. If the solution to the differential equation is unique there are only two trajectories passing through the origin for the two values  $\pm N$  of the control. If the differential equation is of second order, these two trajectories completely separate the space (phase plane for second order systems). If we can geometrically describe these trajectories in terms of the state variables we will have our control function [2,3]. This is best seen by examples.



### III Examples of synthesis of the control functions

#### Example 1. Two Null Roots

Given:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (5)$$

and

$$\underline{x}(0) = \underline{c}, \quad \underline{x}(T) = \underline{0}$$

Find  $u$  such that

$$x_3 = \int_0^T dt \quad (\text{minimum}) \quad (6)$$

where  $u$  is constrained,  $|u| \leq N$ .

Solution:

The Hamiltonian is

$$H = p_1 x_2 + p_2 u + p_3 \quad (7)$$

The maximum of  $H$  minimizes (6). This is achieved with

$$u = N \operatorname{sgn} p_2 \quad (8)$$





The adjoint variable,  $p_2$ , may be heuristically related to a negative time solution from the origin. Letting  $\tau = -t$  we have

$$x_1 = \frac{N\tau^2}{2} \quad (9)$$

and

$$x_2 = -N\tau \quad (10)$$

Eliminating the parameter  $\tau$  gives

$$x_1 = \frac{x_2^2}{2N} \quad (11)$$

This is the equation for two parabolas with  $u = \pm N$ . Of interest are those parts of the parabolas which are associated with trajectories coming into the origin in forward time. Accomplishing this in (11) and using the results as the equation of the control function gives

$$u = -N \operatorname{sgn} \left[ x_1 + \frac{x_2 |x_2|}{2N} \right] \quad (12)$$

This is indeed the optimum minimum time feedback control function and is readily simulated with analog or digital means.

The system with optimum control is shown in familiar block diagram form in Fig. 1. Fig. 2 shows a typical optimum trajectory in the state space.



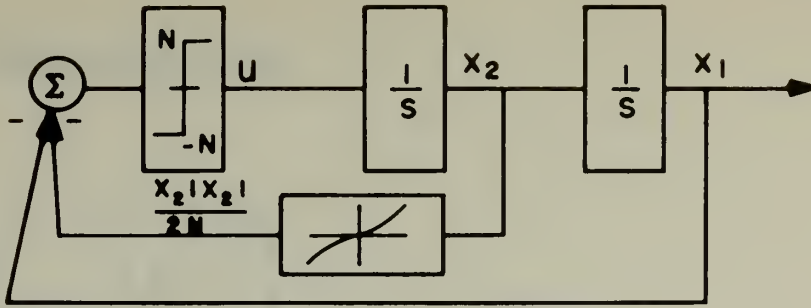


FIG.1 BLOCK DIAGRAM OF SYSTEM WITH MINIMUM TIME FEEDBACK CONTROL.

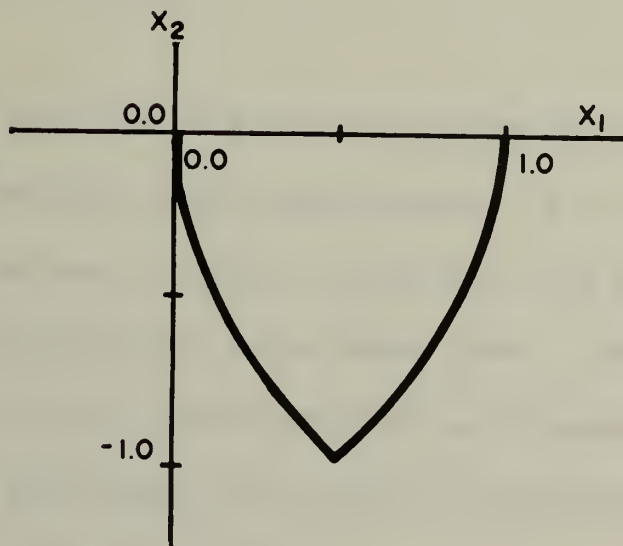


FIG. 2 OPTIMUM TRAJECTORY FOR

$$s^2 X(s) = U(s)$$

$$x_1(0) = 1.0, \quad x_2(0) = 0, \quad U = \pm 1$$



Example 2: A Null and a Real Root

Given:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (13)$$

and

$$\underline{x}(0) = \underline{c}, \quad \underline{x}(T) = \underline{0}$$

Find  $u$  such that

$$x_3 = \int_0^T dt \quad (\text{minimum})$$

where  $u$  is constrained  $|u| \leq N$ .

Solution:

Referring to (7) and (8) which will be of the same form here, we shall move directly to the synthesis of the control function. It is useful to uncouple the state variables to simplify the geometry of our trajectories and to be able to treat the states in an orthogonal space. This is accomplished by taking a partial fraction expansion on the transfer function describing each state with respect to the control and assigning a new state vector component,  $y_i$ , to each eigenvalue,  $\lambda_i$ . For example let



$$\underline{x} = G\underline{y} \quad (14)$$

where

$$x_1 = \frac{u(s)}{s(s+\alpha)} = g_{11} \frac{u(s)}{s} + g_{12} \frac{u(s)}{s+\alpha} \quad (15)$$

and

$$x_2 = \frac{su(s)}{s(s+\alpha)} = g_{21} \frac{u(s)}{s} + g_{22} \frac{u(s)}{s+\alpha} \quad (16)$$

giving

$$G = \begin{bmatrix} 1/\alpha & -1/\alpha \\ 0 & 1 \end{bmatrix} \quad (17)$$

and

$$G^{-1} = \begin{bmatrix} \alpha & 1 \\ 0 & 1 \end{bmatrix} \quad (18)$$

Let us solve the two uncoupled first order differential equations in negative time ( $t = -\eta$ ). The first differential equation is

$$\dot{y} = -u \quad (19)$$





which has the solution (for  $u = N$ )

$$y_1 = y_1(0) - N\tau \quad (20)$$

Next the adjoint variable is used as an integrating factor to make the differential equation exact. We have

$$\dot{y}_2 = \alpha y_2 - u \quad (21)$$

Multiplying by the adjoint,  $p(\tau) = p_0 e^{-\alpha\tau}$  and cancelling  $p_0$  gives

$$\frac{d}{d\tau}(e^{-\alpha\tau} y_2) = -e^{-\alpha\tau} u \quad (22)$$

Integrating

$$e^{-\alpha\tau_1} y_2 - y_2(0) = - \int_0^{\tau_1} e^{-\alpha\tau} u(\tau) d\tau \quad (23)$$

or

$$y_2 = y_2(0) e^{\alpha\tau_1} - \int_0^{\tau_1} e^{\alpha(\tau_1 - \tau)} u(\tau) d\tau \quad (24)$$

Note that we are considering  $u(\tau)$  to be constant over the interval. Finally we have (for  $u = N$ )



$$y_2 = y_2(0)e^{\alpha\tau} + \frac{N}{\alpha}(1 - e^{\alpha\tau}) \quad (25)$$

The forward time solution for  $y_2$  is

$$y_2(s) = \frac{y_2(0)}{s+\alpha} + \frac{N}{s(s+\alpha)} \quad (26)$$

or

$$y_2(t) = y_2(0)e^{-\alpha\tau} + \frac{N}{\alpha}(1 - e^{-\alpha\tau}) \quad (27)$$

In negative time from the origin we have

$$y_1 = -N\tau \quad (28)$$

and

$$y_2 = \frac{N}{\alpha}(1 - e^{\alpha\tau}) \quad (29)$$

Again eliminating the parameter  $\tau$  we obtain the resulting control

$$u = -N \operatorname{sgn} \left[ y_1 - \frac{N}{\alpha} (\operatorname{sgn} y_2) \ln \left( 1 + \frac{\alpha}{N} |y_2| \right) \right] \quad (30)$$

The control function described in (30) is immediately ready for digital simulation. A two term approximation of the log function is taken for analog synthesis.



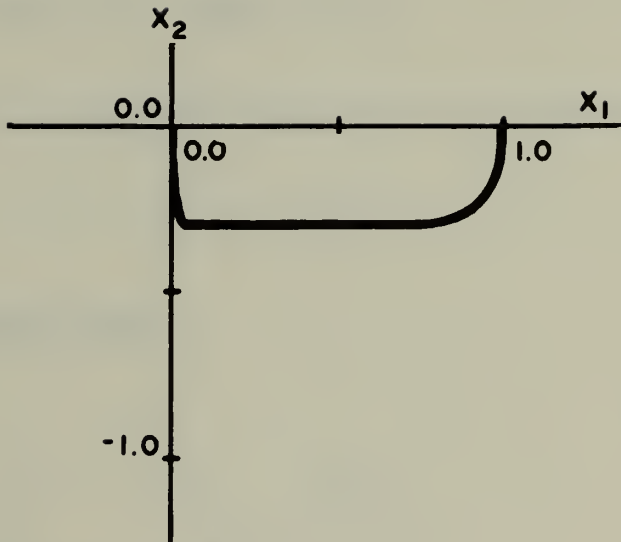


FIG. 3 OPTIMUM TRAJECTORY FOR

$$S(S+3.5) X(S) = U(S)$$

$$x_1(0) = 1.0, \quad x_2(0) = 0, \quad U = \pm 1$$



$$u = -N \operatorname{sgn} \left[ y_1 - y_2 + \frac{\alpha}{2N} y_2 |y_2| \right] \quad (31)$$

which in terms of the  $x$  variables is

$$u = -N \operatorname{sgn} \left[ x_1 + \frac{1}{2N} x_2 |x_2| \right] \quad (32)$$

Figure 3 shows a typical optimum trajectory in the state space.

### Example 3: Two Real Roots

Given:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -\gamma\beta & -\gamma-\beta \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (33)$$

and

$$\underline{x}(0) = \underline{c}, \quad \underline{x}(T) = \underline{0}$$

Find  $u$  such that

$$x_3 = \int_0^T dt \quad (\text{minimum})$$

Where  $u$  is constrained,  $|u| \leq N$ .





Solution:

As before we will uncouple the system

$$x_1 = \frac{u(s)}{(s+\gamma)(s+\beta)} = g_{11} \frac{u(s)}{(s+\gamma)} + g_{12} \frac{u(s)}{(s+\beta)} \quad (34)$$

and

$$x_2 = \frac{su(s)}{(s+\gamma)(s+\beta)} = g_{21} \frac{u(s)}{(s+\gamma)} + g_{22} \frac{u(s)}{(s+\beta)} \quad (35)$$

giving

$$\underline{x} = G\underline{y}$$

where

$$G = \begin{bmatrix} \frac{1}{\beta - \gamma} & \frac{1}{\gamma - \beta} \\ \frac{\gamma}{\gamma - \beta} & \frac{\beta}{\beta - \gamma} \end{bmatrix} \quad (36)$$

and

$$G^{-1} = \begin{bmatrix} \beta & 1 \\ \gamma & 1 \end{bmatrix} \quad (37)$$



The solution in negative time from the origin is

$$y_1 = \frac{N}{\gamma} (1 - e^{\gamma\tau}) \quad (38)$$

and

$$y_2 = \frac{N}{\beta} (1 - e^{\beta\tau}) \quad (39)$$

By eliminating the parameter  $\tau$ , from (38) and (39) an expression for the control function is obtained. For the case of  $\gamma < \beta$ , this expression is

$$u = -N \operatorname{sgn} \left[ \frac{N}{\gamma} (\operatorname{sgn} y_1) \ln \left( 1 + \frac{\gamma}{N} |y_1| \right) - \frac{N}{\beta} (\operatorname{sgn} y_2) \ln \left( 1 + \frac{\beta}{N} |y_2| \right) \right] \quad (40)$$

The control above is readily achieved by digital means. A quadratic approximation to the log function is readily generated to give control by analog means. This is (for  $\gamma < \beta$ )

$$u = -N \operatorname{sgn} \left[ y_1 - \frac{\gamma}{2N} y_1 |y_1| - y_2 + \frac{\beta}{2N} y_2 |y_2| \right] \quad (41)$$

which in terms of the  $x$  variables is

$$u = -N \operatorname{sgn} \left[ (\beta - \gamma)x_1 - \frac{\gamma}{2N} (\beta x_1 + x_2) |\beta x_1 + x_2| + \frac{\beta}{2N} (\gamma x_1 + x_2) |\gamma x_1 + x_2| \right] \quad (41')$$



It should be noted here that whenever the exact control function is approximated, the possibility of chatter motions exist. This is true for controls described in (31), (32), (41), and (49). The coefficient on the quadratic term may be adjusted to eliminate this [ 5 ].

In systems where the ratio of the real roots is an integer or nearly so, a simple optimum control may be generated. The equivalent of (38) and (39) expressed as function of time are

$$\tau = \frac{N}{\gamma} \ln \left( 1 + \frac{\gamma}{N} |y_1| \right) \quad (42)$$

or

$$\tau = \ln \left( 1 + \frac{\gamma}{N} |y_1| \right)^{N/\gamma} \quad (43)$$

and similarly

$$\tau = \ln \left( 1 + \frac{\beta}{N} |y_2| \right)^{N/\beta} \quad (44)$$

Now let us use the arguments of the logarithms in (43) and (44) to obtain the control function. This is (for  $\gamma < \beta$ )

$$u = -N \operatorname{sgn} \left[ (\operatorname{sgn} y_1) \left( 1 + \frac{\gamma}{N} |y_1| \right)^{\beta/\gamma} - (\operatorname{sgn} y_2) \left( 1 + \frac{\beta}{N} |y_2| \right) \right] \quad (45)$$



where  $\beta/\gamma$  is an integer or nearly so. Now simply use the binomial expansion in (45) to obtain

$$u = -N \operatorname{sgn} \left\{ (\operatorname{sgn} y_1) \left[ |y_1| + \frac{(\beta-\gamma)}{2N} |y_1|^2 + \dots \right] - y_2 \right\} \quad (46)$$

For example, let  $\gamma = 1/2$ ,  $\beta = 1$ . The optimum control which yields identical switching to the function given in (40) is

$$u = -N \operatorname{sgn} \left[ y_1 + \frac{y_1}{4N} |y_1| - y_2 \right] \quad (46')$$

Figure 4 shows an optimum trajectory using either (40) or (46').

#### Example 4: Two Imaginary Roots

Given:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (47)$$

and

$$\underline{x}(0) = \underline{c}, \quad \underline{x}(T) = \underline{0}$$

Find  $u$  such that





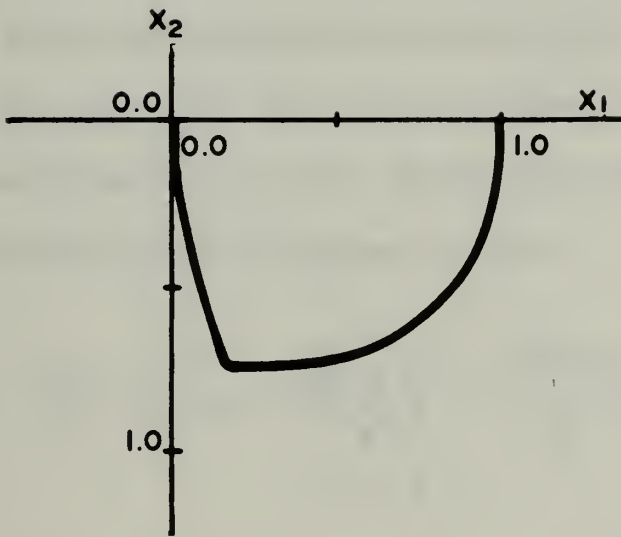


FIG. 4 OPTIMUM TRAJECTORY FOR

$$(s+1)(s+0.5) X(s) = U(s)$$

$$x_1(0) = 1.0, \quad \dot{x}_1(0) = 0, \quad U = \pm 1$$



$$x_3 = \int_0^T dt \quad (\text{minimum})$$

where  $u$  is constrained,  $|u| \leq N$ .

Solution:

It is noted that the zero trajectories are half circles with centers at  $(\pm \frac{iN}{\omega^2}, 0)$ . It can be shown [1,4], that the optimum switching is a set of half circles arrayed along the  $x_1$ -axis. The optimum control function may then be described with the aid of a fourier series as

$$u = -N \operatorname{sgn} \left[ \frac{x_2}{\omega} + (\operatorname{sgn} x_1) \sqrt{\left| \sum_{k=1,3,\dots} \frac{3^2}{\pi^2 k^3} \sin \frac{\omega^2 k \pi x_1}{2N} \right|} \right] \quad (48)$$

$k = 1, 3, \dots$

A simple and effective approximation for analog simulation is given by

$$u = -N \operatorname{sgn} \left[ \frac{x_2}{\omega} + 1.016 (\operatorname{sgn} x_1) \sqrt{\left| \sin \frac{\pi \omega^2}{2N} x_1 \right|} \right] \quad (49)$$

The system is described in Figure 5, with typical trajectory using the above control function.

Example 5: Control of a second order nonlinear system (Van der Pol).

Given:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & (1 - x_1^2) \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (50)$$



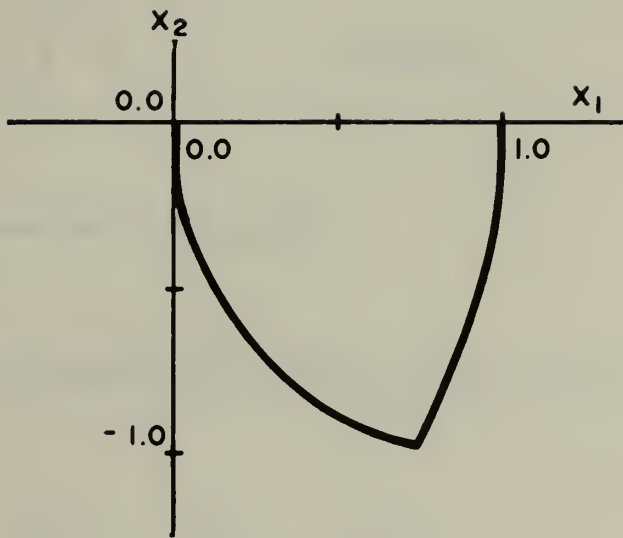


FIG. 5 OPTIMUM TRAJECTORY FOR

$$(s^2+1)X(s) = U(s)$$

$$x_1(0) = 1.0, \quad x_2(0) = 0, \quad u = \pm 1$$

Fig 5



and  $\underline{x}(0) = \underline{c}$  ,  $\underline{x}(T) = \underline{0}$  .

Find  $u$  such that

$$x_3 = \int_0^T dt \quad (\text{minimum})$$

where  $u$  is constrained,  $|u| \leq N$ .

Solution:

The problem is analyzed by taking the solution to the linear problem

[ 3 ],

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (51)$$

and simply let

$$\zeta = -\frac{1}{2} (1 - \bar{x}_1^2)^* \quad (52)$$

For systems with complex roots the states are not completely uncoupled.

However, a transformation is applied which places the real part of the

roots on the main diagonal. Thus (51) is transformed with

$$\underline{x} = \begin{bmatrix} 1 & -\zeta/\nu \\ 0 & 1/\nu \end{bmatrix} \underline{y} \quad (53)$$

\*Here  $\bar{x}_1$  denotes a weighted mean of its present and desired final states.





and

$$\underline{y} = \begin{bmatrix} 1 & \zeta \\ 0 & \nu \end{bmatrix} \underline{x} \quad (54)$$

where

$$\nu = \sqrt{|1 - \zeta^2|}$$

giving

$$\dot{\underline{y}} = \begin{bmatrix} -\zeta & \nu \\ -\nu & -\zeta \end{bmatrix} \underline{y} + \begin{bmatrix} \zeta \\ \nu \end{bmatrix} u \quad (55)$$

The negative time solution from the origin is

$$y_1 = -1 + e^{\zeta\tau} \cos \nu\tau \quad (56)$$

$$y_2 = e^{\zeta\tau} \sin \nu\tau \quad (57)$$

for

$$u = +1 \quad \text{and} \quad 0 < \nu\tau < \pi.$$



After eliminating the parameter  $\tau$  from (56) and (57), an expression

for the control function is obtained in the neighborhood of the origin

( $|x_1| < 1 + \alpha^2$ , where  $\alpha = e^{\zeta\pi/2\nu}$ ),

$$u = -\text{sgn} \left\{ (x_1^2 + 2\zeta x_1 x_2 + x_2^2) \text{sgn } x_2 + 2(x_1 + \zeta x_2) + (e^z - 1) \text{sgn } x_1 \right\} \quad (58)$$

where

$$z = \frac{2\zeta}{\nu} \arccos \frac{1 - (x_1 + \zeta x_2) \text{sgn } x_1}{\sqrt{1 - 2(x_1 + \zeta x_2) \text{sgn } x_1 + x_1^2 + 2\zeta x_1 x_2 + x_2^2}} \quad (59)$$

For  $|x_1| \geq 1 + \alpha^2$ , we may use a linear approximation to the optimum switching curve. This is simply

$$u = -\text{sgn} \left[ \nu x_2 + \frac{(\alpha^2 - 1)}{2\alpha} (x_1 + \zeta x_2) + \frac{(\alpha^2 + 1)}{2\alpha} \text{sgn } x_1 \right] \quad (60)$$

The control functions above were checked for various initial conditions with digital computer simulation.

A quadratic approximation for the small signal control of (58) is

$$u = -\text{sgn} \left[ 2(x_1 + \zeta x_2) - x_1 |x_1| - 2\zeta x_1 x_2 + x_2 |x_2| \right] \quad (61)$$

A control of the form of (58) and (60) is demonstrated in Figure 6. Also a typical uncontrolled limit cycle is shown.



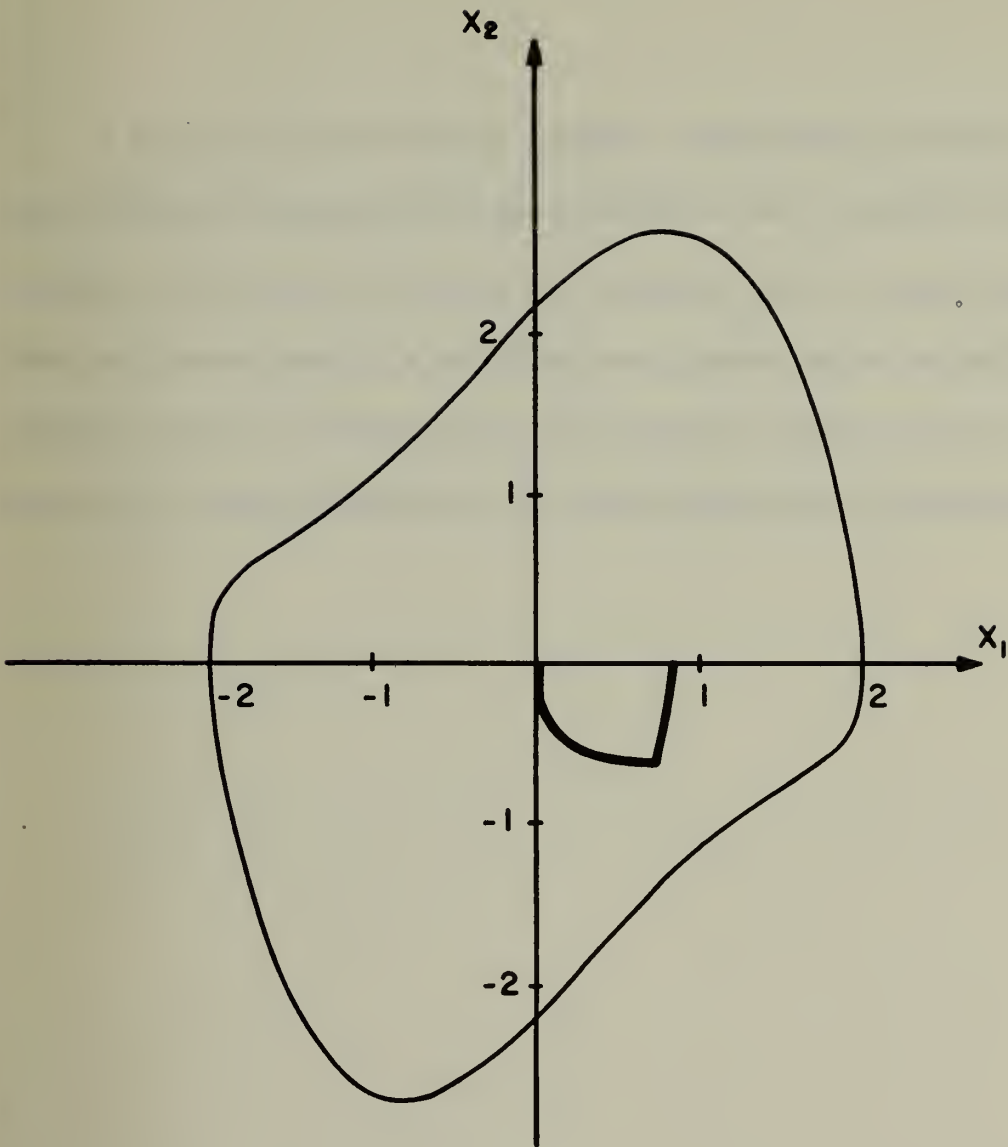


FIG. 6 CONTROLLED AND  
UNCONTROLLED TRAJECTORIES  
FOR VAN DER POL EQUATION  
WITH  $U = \pm 1$



#### IV CONCLUSIONS

It is hoped that this rather compact compilation of optimum and quasi-optimum minimum time control functions may be useful to the practicing engineer. The results of control of a nonlinear system using linearization about each state point in a trajectory were reassuring to the doubtful. The solutions were all checked against true optimum trajectories by coming from the origin in negative time with the control satisfying the Hamiltonian (2).



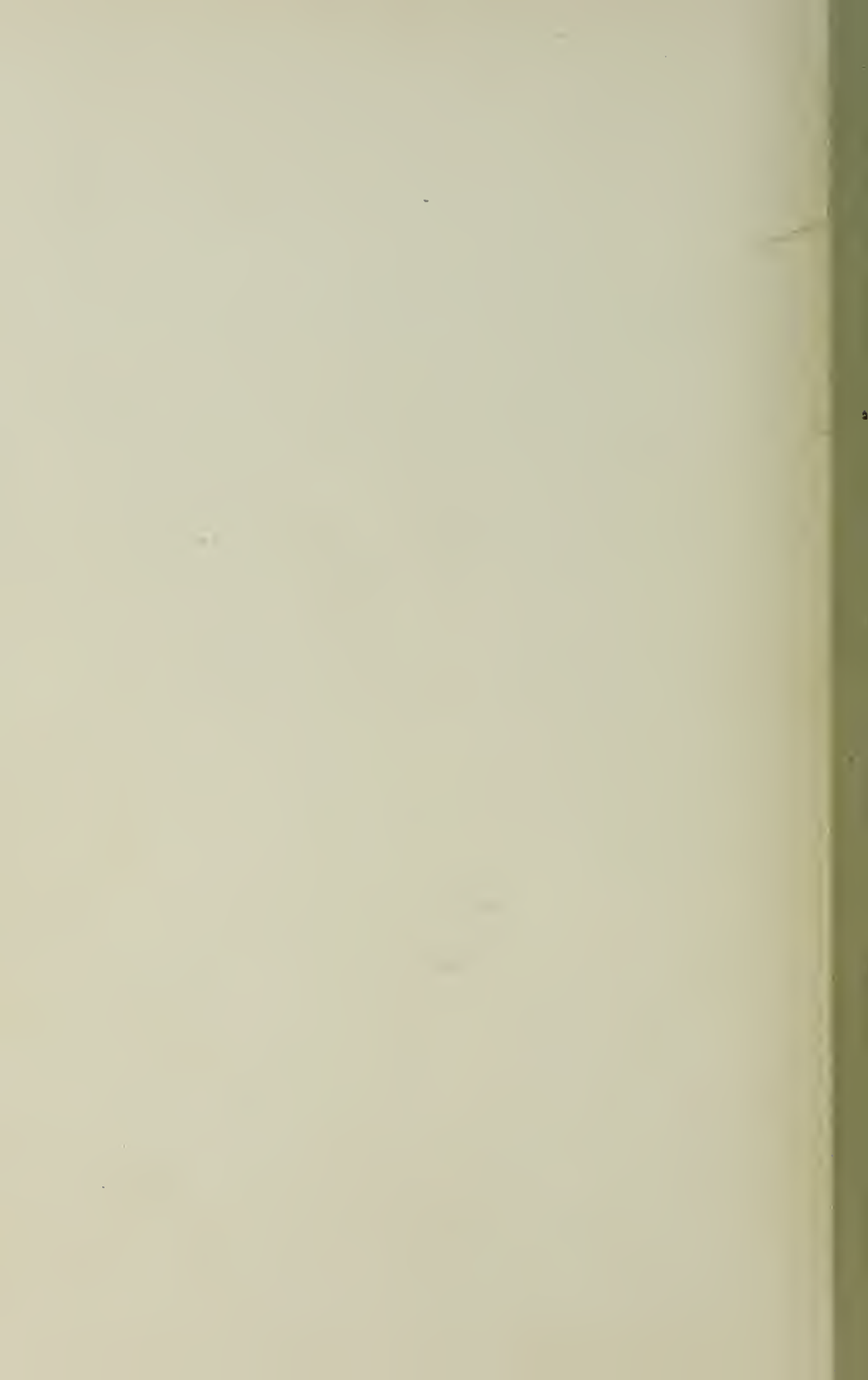


## REFERENCES

1. Pontrjagin, L. S., Boltjanski, V. G., Gamkrelidze, R. V., and Mishcenko, E. F., "The Mathematical Theory of Optimal Processes," 1962, New York: Interscience.
2. "Flugge-Lotz, I., and Titus, H. A., "The Optimum Response of Full Third-Order Systems with Contactor Control," Journal of Basic Engineering, Transactions of the ASME, December, 1962, pp. 554 - 558.
3. Titus, H. A., "Optimum and Quasi-Optimum Control of Third - and Fourth - Order Systems," PH.D. Thesis, Division of Engineering Mechanics, Stanford University, Stanford, California, October, 1962.
4. Bushaw, D., "Optimal Discontinuous Forcing Terms," Contributions to the Theory of Nonlinear Oscillations, 1958, Princeton: Princeton University Press.
5. Hubbs, R. A., "Adaptive Quasi-Optimum Switching Surfaces for Contractor Control Systems," Technical Report No. 2106-4, Stanford Electronics Laboratories, Stanford University, Stanford, California, April, 1962.







29 DEC 1993

MAR 29 1985

1

Keep this card in the book pocket  
Book is due on the latest date stamped

TA7

.U6

no.38

Titus

Optimal control systems,  
Part II (Synthesis).

69395

genTA 7.U6 no.38

Optimal control systems.



3 2768 001 61406 8

DUDLEY KNOX LIBRARY